Block 1

Preparing the trip

CHAPTER 1

Geometry in the plane and space. Matrices of order 2 and 3

The content of this topic is a presentation of everything that will be studied in the course, to fix a bit the ideas and, above all, to come back to it as you progress through the later topics. All the examples that will be seen in this chapter can be taken up as we progress. It should not be forgotten that the objective of this course is to start thinking about mathematics and this is done from the abstraction of already known ideas that will shape new theories and work environments detailed by the conditions that we impose, taken from practice. Section 2 is dedicated to an in-depth study of matrices, emphasizing elementary matrices that will be very useful in the next two topics. We finish the topic with the study and properties of the determinants, a fundamental and very useful tool for many demonstrations and justification of the results.

1. Geometry in the plane and in space. Objects and transformations.

We begin by studying the Cartesian plane whose points are pairs of numbers that can also be seen as the solutions of systems of equations with two unknowns. We start by seeing the difference of studying the solutions in the different sets of numbers. The natural numbers are represented by:

$$\mathbb{N} = \{0, 1, 2, \ldots\}$$

The integer numbers are represented by:

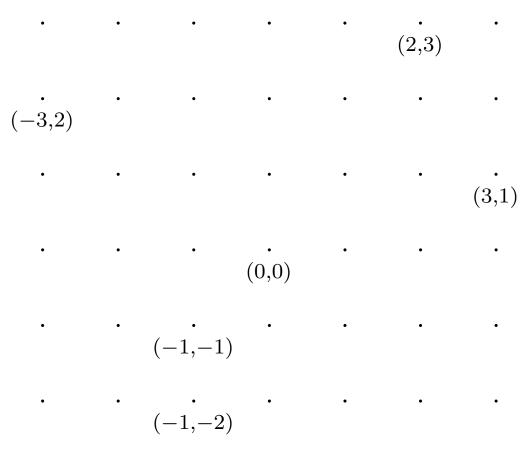
$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

The rational numbers are represented by

$$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}\}$$

And the real numbers represented by $\mathbb{R}$, which covers all points on a line.

1.1. The Cartesian plane. If we consider the set of points with integer coordinates ($\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$) we can draw it like:



In this set the following system has no solution:

$$\left.\begin{array}{rcl} 2x + y &=& 0 \\ x - y &=& 1 \end{array}\right\}$$

To solve the system with Mathematica:

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Solve[{2 x+y==0, x-y==1},{x,y}]
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Since its solution is $(\frac{1}{3}, \frac{-2}{3})$. However, it has a solution when we can work on $\mathbb{Q}$.

We now consider the system of equations:

$$\left.\begin{array}{rcl} x^2+y^2 & = & 1 \\ x-y & = & 0 \end{array}\right\}$$

And we calculate their solutions:

Solve [{xˆ 2+yˆ 2==1,x-y==0},{x,y}]

we get $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ y $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, we can verify that they are not in $\mathbb{Q}$.

EXERCISE 1. [1] *Prove that the number $\sqrt{2}$ is not in $\mathbb{Q}$. (The most convenient thing is to search on the Internet). Search Wikipedia: square root of 2, Proof by infinite descent.*

EXERCISE 2. [2] *TO DEEPEN: Study the curious relationship between $\mathbb{Q}$ and $\mathbb{R}$, in terms of* ***numerability*** *and* ***density****. Study how the decimal expression of the elements of $\mathbb{Q}$ and $\mathbb{R}$ is. More specifically, investigate that $\mathbb{Q}$ is dense in $\mathbb{R}$ and yet $\mathbb{Q}$ is countable but $\mathbb{R}$ is not.*

The following instruction gives us an approximation to 40 decimal places of the real numbers $\sqrt{2}$, π and for the rational number $\frac{231}{312}$:

N[Sqrt[2],40], N[Pi,40], N[231/312,40], ...

Here we impose our first condition, we will work with the field of real numbers, or in general in a field where we can calculate the inverse of a non-zero number. The best known fields are $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. We will give the definition later.

Within the sets of $\mathbb{R}^2$, we will focus on straight lines and points, since these elements will be the simplest to study and generalize to larger dimensions.

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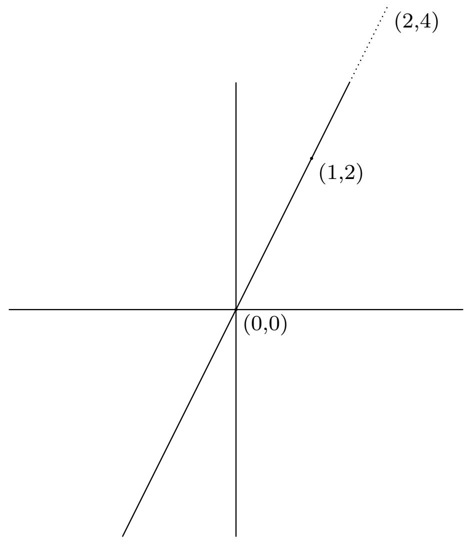
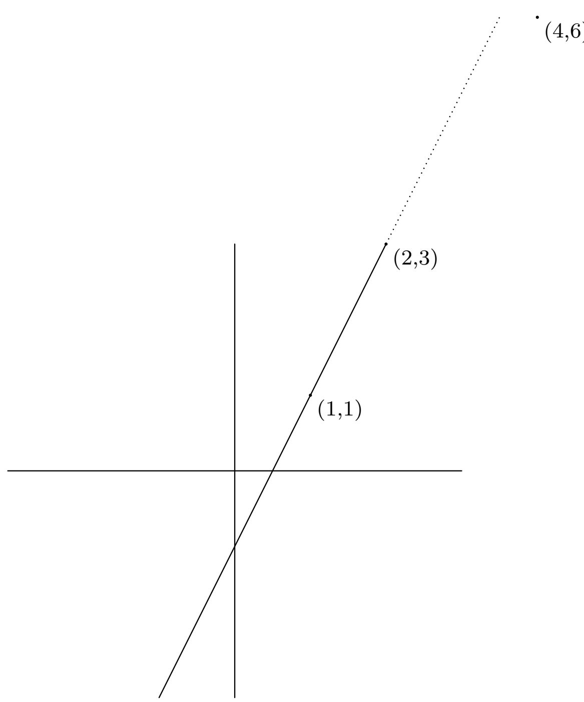
[1] With this exercise one can evaluate UAL1 competency and the Use of Information Technologies competency that is not evaluated in this subject

[2] With this exercise one can evaluate UAL1 and CB1 competencies and UIT competency that is not evaluated in this subject

1.2. Lines that pass through the origin and lines that do not.
Let's draw the line that passes through $(0,0)$ and $(1,2)$:



For which other points of integer coordinate points does the line pass? Let us draw the line that passes through $(1,1)$ and $(2,3)$. What other integer coordinate points does it pass through?



Why does the first line pass through $(2,4) = 2(1,2)$ and the second line not pass through $(4,6) = 2(2,3)$? The equations of both lines are:

$$\begin{aligned} 2x - y &= 0 \\ 2x - y &= 1 \end{aligned}$$

What are both lines like? Try to solve the system formed by the two equations.

To draw with Mathematica:

Plot[2*x-y=0,2*x-y=1]

EXERCISE 3. [3] *All lines in the plane are of the form $y = mx+n$ where m and n are variables. Prove, without using any formula from the views in high school, that there is a single straight line that passes through $p = (p_1, p_2)$ and $q = (q_1, q_2)$. Calculate it explicitly for the points $p = (1,2)$ and $q = (3,-1)$.*

Note for the exercise. You can start by doing $p = (1,2)$ and $q = (3,-1)$, and then, in general.

1.3. Matrices. Multiplication and symmetries. We know from high school that the formula for multiplying matrices two by two is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} ax+bz & ay+bt \\ cx+dz & cy+dt \end{pmatrix}$$

In Mathematica, we can multiply matrices by means of:

{{a,b},{c,d}}.{{x,y},{z,t}}

In this brief section, we see how matrix multiplication can reflect certain symmetries and model plane transformations. We are extending these ideas to higher dimensions.

How do we move from $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\begin{pmatrix} b & a \\ d & c \end{pmatrix}$? And how do we get $\begin{pmatrix} c & d \\ a & b \end{pmatrix}$?

[3]With this exercise one can evaluate UAL1 and CE2 competencies

In both cases, everything is reduced to multiplying to the right or left by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

by using Mathematica:

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{{a,b},{c,d}}.{{0,1},{1,0}}
{{0,1},{1,0}}.{{a,b},{c,d}}
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Obtaining:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

Other transformations and multiplication. Matrix multiplication also serves to represent plane transformations. For example, to obtain the rotation of 90^o in $\mathbb{R}^2$ we use the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ as follows:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

With Mathematica:

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{{0,-1},{1,0}}.{{1},{1}}
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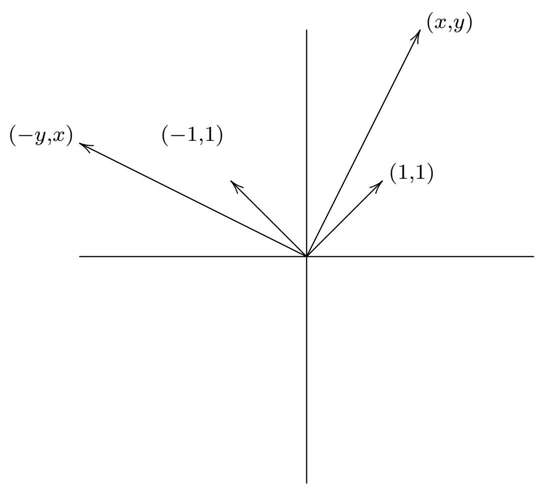
In general, the transformation of (x, y) is $(-y, x)$.



Reflection with respect to the OY axis is given by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ while the homothety of ratio k is obtained with the matrix: $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.

EXERCISE 4. [4] *Enter these examples into Mathematica.*

INVENTING TRANSFORMATIONS. Following the above examples, we have that every 2×2 matrix gives rise to a transformation of the plane. For example:

EXAMPLE 1.

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

{{1,1},{1,2}}.{{2},{3}}

EXAMPLE 2. *What happens with the transformation*

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}?$$

Where is the element $(1,0)$ *mapped? and the element* $(0,1)$*? How are all the elements transformed?*

{{1,2},{1,2}}.{{1},{0}}
{{1,2},{1,2}}.{{0},{1}}

We can determine some equations for these transformations.

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+2y \end{pmatrix}$$

{{1,1},{1,2}}.{{x},{y}}

That is, the transformation of (x, y) is $(x+y, x+2y)$. So, to know if, for example, the vector $(3,5)$ is the transform of a vector, it is enough to solve the system:

$$\left.\begin{array}{rcl} x+y & = & 3 \\ x+2y & = & 5 \end{array}\right\}$$

Solve[{x+y==3,x+2.y==5},{x,y}]

[4]With this exercise one can evaluate CB1 competency and UIT competency that is not evaluated in this subject

Whose solution is $(1, 2)$. For the second transformation, it would be that the elements obtained after the transformation must be of the form: $(x + 2y, x + 2y)$, that is, the two coordinates must be equal.

But we could also do it the other way around, that is, we can build transformations that take given points to given points. For example, find a transformation that takes $(1, 0)$ to $(3, 4)$ and $(0, 1)$ to $(5, 1)$. To do this, let us observe that:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

{{a,b},{c,d}}.{{1},{0}}

and so:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

{{a,b},{c,d}}.{{0},{1}}

So, the required transformation could be made using the matrix:

$$\begin{pmatrix} 3 & 5 \\ 4 & 1 \end{pmatrix}$$

At the moment we can only construct transformations giving the transforms of $(1, 0)$ and $(0, 1)$. We will learn to do it in general.

1.4. Maps. The usual way in mathematics to write transformations is through maps between sets. Let us go back to the transformation of the example 1 and remember that:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x + 2y \end{pmatrix}$$

That is, we can think of this multiplication as transforming a point (x, y) in $\mathbb{R}^2$ into the point $(x + y, x + 2y)$ in $\mathbb{R}^2$. If we call that transformation f, in mathematics we write it as: $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined as $(x, y) \mapsto (x + y, 2x + y)$.

With this new notation, we have to:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

will be written as $f(1, 0) = (1, 1)$ and $f(0, 1) = (1, 2)$. The transformed element $(1, 1)$ is called **image** of $(1, 0)$. We know that the first column of the matrix is the transform of $(1,0)$ and the second column is the transform of $(0,1)$. But we can observe that:

$$\begin{pmatrix} x+y \\ x+2y \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

From this fact we can draw two conclusions, the first is that:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix}$$

And, on the other hand: $f(x,y) = xf(1,0) + yf(0,1)$.

For the map f, you can define the return transformation, that is, the **inverse transformation** which would be:

$$\begin{aligned} f^{-1}\colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ (u,v) &\mapsto (2u-v,-u+v) \\ (5,8) &\mapsto (2,3) \end{aligned}$$

If we compose f and f^{-1}:

$$\mathbb{R}^2 \xrightarrow{\quad f \quad} \mathbb{R}^2 \xrightarrow{\quad f^{-1} \quad} \mathbb{R}^2$$

$$(u,v) \longmapsto (2u-v,-u+v)$$

$$(x,y) \longmapsto (x+y,x+2y) \longmapsto (x,y)$$

$$(2,3) \longmapsto (5,8) \longmapsto (2,3)$$

This is called **maps composition** which is nothing more than transforming through one map and then through another. For example, if we first apply the transformation of Example 1 and then that of Example 2, which we will call g, we have:

$$\mathbb{R}^2 \xrightarrow{\quad f \quad} \mathbb{R}^2 \xrightarrow{\quad\quad g \quad\quad} \mathbb{R}^2$$

$$(u,v) \longmapsto (u+2v, u+2v)$$

$$(x,y) \longmapsto (x+y, x+2y) \longmapsto (3x+5y, 3x+5y)$$

$$(2,3) \longmapsto (5,8) \longmapsto (21,21)$$

As we saw before, the map f has the peculiarity of having an inverse. Each point has one image and only one, and all images have one preimage and only one. However, the map g does not have this property, for example $g(1,1) = (3,3) = g(3,0)$. And any element that does not have the same two coordinates will not have a preimage.

Exercise 5. [5] *Find other points that, using the previous map g, have the same image.*

The determinant: a help. We also know from high school that the determinant of a two-by-two matrix is:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

The command in Mathematica for the determinant is:

Det[{{a,b},{c,d}}]

With this definition, it can be seen that if we consider the determinants of the matrices of the previous examples, the determinant of the matrix of the example (1) is different from zero while the second determinant cancels out. This will be very interesting data to discern the transformations, and that we will study in more depth.

Thus, while the first has an inverse transformation that recovers the plane as it was, with the second transformation this is impossible. The

[5] With this exercise one can evaluate CB1 competency and the UIT competency that is not evaluated in this subject

matrix of the inverse transformation of the example (1) is: $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ as:

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x+y \\ x+2y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

To calculate the inverse matrix in Mathematica enter:

Inverse[{{1,1},{1,2}}]

EXERCISE 6. [6] *Check using Mathematica that:*

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and also:

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1.5. Systems of equations. Let's consider systems of equations:

$$\left.\begin{array}{rcl} 2x+2y & = & 1 \\ x-y & = & 0 \end{array}\right\} \quad \left.\begin{array}{rcl} 4x+4y & = & 2 \\ x-y & = & 0 \end{array}\right\}$$

$$\left.\begin{array}{rcl} x-y & = & 0 \\ 2x+2y & = & 1 \end{array}\right\} \quad \left.\begin{array}{rcl} 2x+2y & = & 1 \\ 3x+y & = & 1 \end{array}\right\}$$

$$\left.\begin{array}{rcl} x & = & \frac{1}{4} \\ y & = & \frac{1}{4} \end{array}\right\}$$

They all have the same solution: $x = y = \frac{1}{4}$, which comes out immediately from the last system.

EXERCISE 7. [7] *Check it using Mathematica.*

DEFINITION 1. We say that two systems of equations are **equivalent** if they have the same solutions.

We will discuss systems of equations in the next section. Here we will only focus on the following exercise, which is simpler for the case of two equations with two unknowns, although we will see and demonstrate it in general.

[6] With this exercise one can evaluate CB2 competency

[7] With this exercise one can evaluate CB2 competency

Exercise 8. [8] *Show Cramer's rule without using the inverse matrix. That is, given a system of equations with a unique solution,*

$$\left.\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 & = & b_1 \\ a_{21}x_1 + a_{22}x_2 & = & b_2 \end{array}\right\},$$

prove that this solution can be obtained as:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Exercise 9. [9] *Do the previous exercise using Mathematica.*

1.6. 3D-Space. Vector planes and lines. Systems of equations. Gauss-Jordan Method. There is a method due to Gauss and improved by Jordan that is used to solve systems of 3 equations with 3 unknowns, but also, as we will study later, for systems of m equations with n unknowns. We will see an example of a system of 3 equations and 3 unknowns and later we will generalise it.

Example 3.

$$\left.\begin{array}{rcl} x + 2y - 3z & = & -16 \\ 3x + y - 2z & = & -10 \\ 2x - 3y + z & = & -4 \end{array}\right\} \to \left.\begin{array}{rcl} x + 2y - 3z & = & -16 \\ -5y + 7z & = & 38 \\ -7y + 7z & = & 28 \end{array}\right\} \to$$

$$\left.\begin{array}{rcl} x + 2y - 3z & = & -16 \\ -5y + 7z & = & 38 \\ -\frac{14}{5}z & = & -\frac{126}{5} \end{array}\right\} \to \left.\begin{array}{rcl} x + 2y - 3z & = & -16 \\ -5y + 7z & = & 38 \\ 14z & = & 126 \end{array}\right\}$$

In the first step, the unknown "x" has been removed from the last equations by adding to both the first multiplied by -3 and by -2 respectively. In the second step, the unknown "y" has been removed from the last equation by adding the second multiplied by $\frac{-7}{5}$. In the last step, multiplied by 5 the last equation.

[8] With this exercise one can evaluate CB2, CE2, and RD1 competencies

[9] With this exercise one can evaluate RD1, and CB1 competencies and the UIT competency that is not evaluated in this subject